Notes: 11.1 Sequences

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7:24 PM

11.1 Sequences

1. Intro

   A sequence is any set of numbers in some order

   $\{a_1, a_2, a_3, \ldots\ A_n\}$

   $\{a_n\}$

   A_n

   a. $\{1,2,3,4,5,\ldots..n\}$ $\{n\}$ $\{a_n\}$ where $a_n=n$
   b. $\{1/2, 2/3, 3/4, \ldots..n/n+1\ldots..\}$ n=n/n+1
   c. $\{1/2, 1/4, 1/6 \ldots.\ 1/2n, \ldots.\}$ n= 1/2n
   d. $\{3/6, -4/25, 5/125, -6/625, 7/3125\ldots\ n\}$ $n=[(-1)^{n+1}(n+2+)]/5^n$ $^*(-1)^{n+1}$= alternating sequence
   e. $\{1,1,2,3,5,8,\ldots.\}$ $a_n=a_{n-1}+a_{n-2}$ fibonacci series

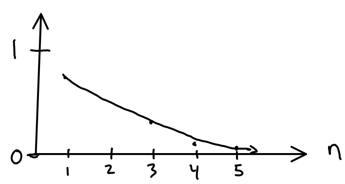
2. Plot

$a_n = \frac{1}{2n}$ $\quad a_4 = \frac{1}{8}$

$a_1 = \frac{1}{2}$ $\quad a_5 = \frac{1}{10}$

$a_2 = \frac{1}{4}$

$a_3\ \frac{1}{6}$



Consider $a_n = \frac{n}{n+1}$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$ $\quad a_6 = \frac{6}{7}$

$a_2 = \frac{2}{3}$ $\quad a_7 = \frac{7}{8}$

$a_3 = \frac{3}{4}$

$a_5 = \frac{5}{6}$ $\quad$ so $\lim_{n\to\infty} \frac{n}{n+1} =$

alternatively $\div$ top & bottom by n

$a_n = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \longrightarrow \frac{1}{1+0} = 1$

techically

when a_n has a limit

for every $\varepsilon > 0$ $\exists$ an $N \ni |a_n - L| < \varepsilon$ for all $n > N$

☆ $\exists$ there exists

$\ni$ such that

$\forall$ for all

$\therefore$ therefore

Also, we may be able to replace a_n with a function $f(x)$

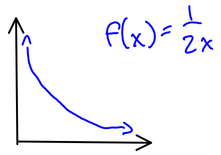
Also, we may be able to replace a_n with a function $f(x)$ such that $f(n) = a_n$

$$\{a_n\}_{n=1}^{\infty} = \{f(x)|_n\}_{n=1}^{\infty}$$

ex) $a_n = \frac{1}{2n}$

$f(x) = \frac{1}{2x}$

$f(n) = \frac{1}{2n}$



3. Properties
   a. If $\{a_n\} \to P$
      $\{b_n\} \to Q$

$$\lim_{n\to\infty}\left(\frac{1}{2n} - \frac{n}{n+1}\right) = \lim\left(\frac{1}{2n}\right) - \lim\left(\frac{n}{n+1}\right) = 0 - 1 = -1$$

   b. Lim $\{ca_n\}$ = clim$\{a_n\}$

   c. Lim $\{a_n \times b_n\}$ = lim $\{a_n\}$ x lim$\{b_n\}$ = P x Q

   d. Lim $\{a_n/b_n\}$ = Lim$\{a_n\}$/Lim$\{b_n\}$

$$\lim_{n\to\infty}\{a_n^p\} = \lim_{n\to\infty}\{a_n\}^p = P^p$$

Theorem: Squeeze Theorem: For $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ if $a_n<b_n<c_n$ then if $\{a_n\} \to L$ and $\{c_n\} \to L$ then lim n->infinity $\{b_n\}$ =L

Examples:

ex) $a_n = \frac{n!}{n^n}$

$a_1 = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{2}{9}$

$a_4 = \frac{24}{625}$

$$\frac{n!}{n^n} = \underbrace{\frac{1}{n}}_{0}\underbrace{\left(\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdot\frac{n-1}{n}\right)}_{<1}\frac{n}{n}$$

$$0 < a_n < \frac{1}{n}$$

$\lim_{n\to\infty}\frac{n!}{n^n} = 0$ by squeeze theorem

4. Theorem: $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

5. More definitions
   a. Definition: $\{a_n\}$ is increasing if $a_{n+1}>a_n$
   b. Definition: $\{a_n\}$ is decreasing if $a_{n+1}<a_n$
   c. Definition: $\{a_n\}$ is bounded above if $a_n <$ M for all n
   d. Definition: $\{a_n\}$ is bounded below if $a_n >$P
   e. Definition: $\{a_n\}$ is bounded below and bounded above, then $\{a_n\}$ is a bounded sequence
   f. Theorem: Every bounded monatomic sequence is convergent.

#62
let $a_1=\sqrt{2}$ $a_{n+1} = \sqrt{2+a_n}$

$a_1 = \sqrt{2}$
$a_2 = \sqrt{2+\sqrt{2}}$
$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

Part I: Show the sequence is increasing

1. $n=1 \quad a_1 = \sqrt{2}$
2. assume $n=k$ follows the same (increase)

$a_{k+1} > a_k$

$a_{k+1} + 2 > a_k + 2$

$\sqrt{a_{k+1}+2} > \sqrt{a_k+2}$

$a_{k+2} > a_{k+1}$

3. deduce that $a_{k+1} > a_n$ for all $n \geq 1$

Part II: Boundedness

1. note $a_1 = \sqrt{2} < 3$
2. assume that $a_k < 3$

$a_k + 2 < 3+2$

$\sqrt{a_k+2} < \sqrt{5}$

$a_{k+1} < \sqrt{5}$

$a_{k+1} < \sqrt{5} < 3$

$a_{n+1} < 3$

By theorem, it is bounded and monotonic and is thus convergent

Part II: What is the limit?

We know the limit exists let it be L

$\underset{n\to\infty}{\text{limit}}\ a_n = L$

Note $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n+2}$ <—square both sides

$\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (2+a_n)$

$$= 2 + \lim_{n\to\infty} a_n$$

$$L^2 = 2 + L$$

$$L^2 - L - 2 = 0$$

$$(L+1)(L-2) = 0$$

$$L = \cancel{-1} \text{ or } \boxed{2}$$