Notes 13.3

Wednesday, June 27, 2007
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Dot Product

The dot product (scalar product) of $\vec{v} = \langle a_1,a_2,a_3\rangle$ and $\vec{w} = \langle b_1,b_2 b_3\rangle$ is defined as $\vec{v}\cdot\vec{w} = a_1b_1 + a_2b_2 + a_3b_3$.

Note: The dot product results in a scalar.

In R^2 $\langle a,b\rangle\cdot\langle c,d\rangle = ac + bd$

Example 1:

$\langle 2,4\rangle\cdot\langle 3,-1\rangle = (2)(3) + (4)(-1)$

$=6 -4 = 2$

Example 2:

$(i + 2j -3k)\cdot(4i - 5k) = 4 + 0 +15 = 19$

Properties of Dot Product

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

   proof $\vec{a} = \langle a_1, a_2, a_3\rangle$

   $\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2$

   $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

2. Distributive

   $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{C}$

3. Communatitive

   $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

4. $\vec{0}\cdot\vec{a} = 0$

5. $c(\vec{a}\cdot\vec{b}) = (c\vec{a})\cdot\vec{b} = \vec{a}\cdot(c\vec{b})$

Theorem

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$

Example: If $|\vec{a}| = 4$, $|\vec{b}| = 6$ & angle between them is $\pi/3$, find $\vec{a}\cdot\vec{b}$.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= 4\cdot 6\cos\frac{\pi}{3} = 12$

*Need to know trig functions of 0, $\pi/2$, $\pi/3$, $\pi/6$, $\pi/2$

Corollary

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Example: Find the angle between $\vec{a}\langle 2,2,-1\rangle$ and $\vec{b}\langle 5,-3,2\rangle$

$$\vec{a}\cdot\vec{b} = 10-6-2 = 2$$

$$|\vec{a}| = 3 \qquad |\vec{b}| = \sqrt{25+9+4} = \sqrt{38}$$

$$\cos\theta = \frac{2}{3\sqrt{38}}$$

$$\theta = 83.79$$

Corollary

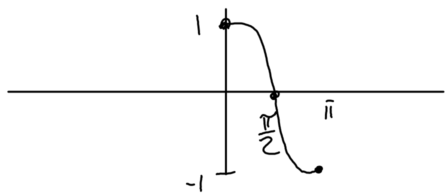
$\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) if and only if their dot product is 0.

$$\vec{a}\perp\vec{b} \iff \theta = \frac{\pi}{2}$$

$$\cos\theta = 0$$

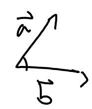
*NOTE $\vec{0}$ is defined to be orthogonal to everything.

Lets consider y=cos θ



$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

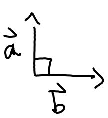
For $0<\theta<\frac{\pi}{2}$



acute ∠

then $\vec{a}\cdot\vec{b}>0$

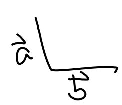
For $\theta>\frac{\pi}{2}$



right ∠

then $\vec{a}\cdot\vec{b}=0$

For $\frac{\pi}{2}<\theta<\pi$



obtuse ∠

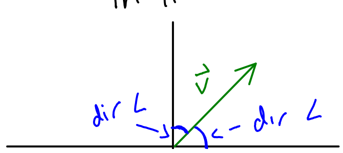
then $\vec{a}\cdot\vec{b}<0$

Example: Show that $\vec{v}$ = 2i + 2j - k and $\vec{w}$ = 5i - 4j +2k are orthogonal.

$\vec{v}\cdot\vec{w} = 10-8-2=0$ So it is orthogonal

Direction Angles

in $\mathbb{R}^2$



The direction angles α, β, γ of vector $\vec{v}$ are the angles $\vec{v}$ makes with the positive x, y, and z axis (ie te angle between $\vec{v}$ and I ,j ,k)

Direction cosines

v = <a₁,a₂,a₃>

$$\cos(\alpha) = \frac{\vec{v} \cdot \vec{i}}{|\vec{v}||\vec{i}|} = \frac{a_1}{|\vec{v}|}$$

$$\cos(\beta) = \frac{a_2}{|\vec{v}|}$$

$$\cos(\gamma) = \frac{a_2}{|\vec{v}|}$$

Something we can do with direction cosines:

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = \frac{a_1^2}{|\vec{v}|^2} + \frac{a_2^2}{|\vec{v}|^2} + \frac{a_3^2}{|\vec{v}|^2} = \frac{1}{|\vec{v}|^2}(a_1^2 + a_2^2 + a_3^3) = \frac{1}{|\vec{v}|^2} \vec{v} \cdot \vec{v} = 1$$

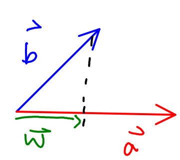
$$\vec{v} = \langle a_1, a_2, a_3 \rangle = \langle |\vec{v}| \cos\alpha, |\vec{v}| \cos\beta, |\vec{v}| \cos\gamma \rangle$$

$$= |\vec{v}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$= \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{unit vector}} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Projections

The projection of b onto a is $\text{proj}_{\vec{a}} \vec{b} = \vec{w}$



$$\star \; |\vec{w}| = |\vec{b}| \cos\theta$$

$$|\vec{w}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \text{comp}_{\vec{a}} \vec{b}$$

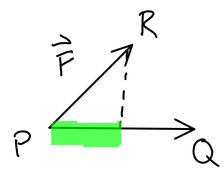
Need to multiply by unit vector in direction of $\vec{a}$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$$

Work

In physics it is Fd (force x distance)

Suppose a constant force is a vector $\vec{F} = \overrightarrow{PR}$ pointing in another direction



$\vec{D} = \overrightarrow{PQ}$ is displacement vector

The force in the direction of $\vec{D}$ is $|\vec{F}|\cos\theta$

$$W = |\vec{F}|\cos\theta\,|\vec{D}| = \vec{F}\cdot\vec{D}$$