# Notes 13.4

Thursday, June 28, 2007
1:30 PM

## The Cross Product

Only defined for 3d vectors
Also called the vector product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

## Determinants

Given a 2x2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex: $$\det \begin{bmatrix} 7 & 10 \\ -3 & 4 \end{bmatrix} = (7)(4) - (-3)(10)$$
$$= 28 + 30 = 58$$

3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Ex: $$\begin{vmatrix} 1 & 7 & 3 \\ 2 & -1 & 0 \\ 1 & -2 & -5 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ -2 & -5 \end{vmatrix} - 7 \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$
$$= 5 - 7(-10) + 3(-4 + 1)$$
$$= 66$$

## Alternate Definition:

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex| $\vec{a} \times \vec{b}$ $\quad \vec{a} = \langle 1, 3, 4 \rangle \quad \vec{b} = \langle 2, 7, -5 \rangle$

$$-15i + 8j + 7k - (6k + 28i - 5j) = -43i + 13j + k$$

$$\vec{a} \times \vec{b} = \left| \begin{matrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{matrix} \right| \begin{matrix} i & j \\ 1 & 3 \\ 2 & 7 \end{matrix}$$

Property: 1) $\vec{a} \times \vec{a} = 0$

2) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
3) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (associativity doesn't hold for cross product)

How to prove a x b is perpendicular to a and b?

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \quad \leftarrow \text{Show these}$$
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

What is the magnitude of the cross product?

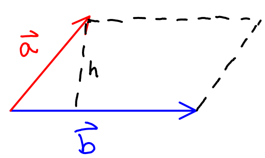
Theorem

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \qquad (0 \le \theta \le \pi)$$

Corollary

$\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parrallel

Ex)




What is the area of the parallelogram

$$\sin\theta = \frac{|\vec{h}|}{|\vec{a}|}$$
$$|\vec{h}| = |\vec{a}| \sin\theta$$

$$|\vec{b}||\vec{h}|$$
$$= |\vec{b}||\vec{a}| \sin\theta$$
$$= |\vec{a} \times \vec{b}|$$

Ex.

Find the area of the triangle with vertices P(1,4,6) Q(-2,5,-1) R(1,-1,1)

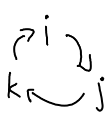
$$\vec{PQ} = \langle -3, 1, -7 \rangle$$
$$\vec{PR} = \langle 0, -5, -5 \rangle$$

Area of $\triangle$ is ½ area of parallelogram
$= \frac{1}{2} |\vec{a} \times \vec{b}|$

Properties

i, j, k

i x j = k
i x k = -j
k x i = j




clockwise = (+)
counterclockwise = (−)

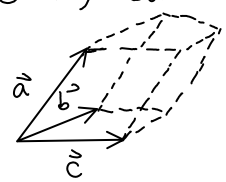
1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2) $(c\vec{a}) \times \vec{b} = \vec{a}(c\vec{b}) \qquad c = \mathbb{R}$
3) $\vec{a}(\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

4) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

5) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

6) $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

☆ $\vec{a}\cdot(\vec{b}\times\vec{c})$ called the scalar tripple product



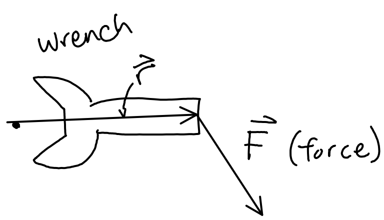

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$ is volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$

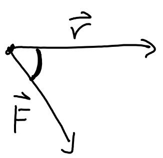
Ex) $\vec{a} = \langle 1,4,-7\rangle \quad \vec{b} = \langle 2,-1,4\rangle \quad \vec{c} = \langle 0,-9,18\rangle$

Show that those are coplanar

One way to check the volume of parallelpided $= 0$

Application Torque

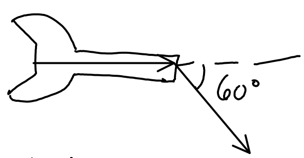



Torque $\vec{\tau} = \vec{r}\times\vec{F}$

Magnitude of torque is $|\vec{\tau}| = |\vec{r}\times\vec{F}| = |\vec{r}||\vec{F}|\sin\theta$

Ex. A bolt is tightened by applying a 40N force to a .25m wrench. Find magnitude of torque.




$|\vec{\tau}| = (.25)(40)\sin 60$

$= 10\cdot\frac{\sqrt{3}}{2} = 5\sqrt{3}$