Notes 13.5

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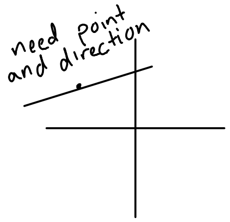
Equations of Lines and Planes

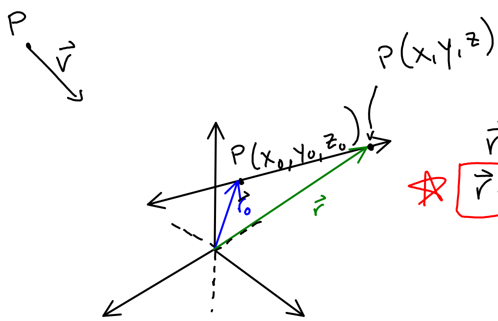
- $\mathbb{R}^2$ to describe a line



$y = mx + b$

$Ax + By = C$

- $\mathbb{R}^3$: need point and direction to describe a line



$\vec{r} - \vec{r}_0 = t\vec{v}$ (for some scalar t)

☆ $\boxed{\vec{r} = \vec{r}_0 + t}$ vector equation of a line

$\vec{v} = \langle a, b, c \rangle \Rightarrow t\vec{v} = \langle ta, tb, tc \rangle$

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$

$= \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$

ie)

☆☆ $\boxed{x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc}$ parametric equation of a line

Solve for t in ☆☆

$t = \boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$ symmetric equations of a line

Ex) line through $(5, 1, 3)$ parallel to $i + 4j - 2k$

Find vector and parametric equations

vector $\vec{r} = \vec{r}_0 + t\vec{v}$

$\vec{r}_0 = \langle 5, 1, 3 \rangle$

$\vec{v} = \langle 1, 4, -2 \rangle$

vector eq. $= \vec{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$

parametric

$x = 5 + t,\ y = 1 + 4t,\ z = 3 - 2t$

Find 2 other points on this line

- Any value of t gives another point on this line

$t = 1$

$x = 6,\ y = 5,\ z = 1 \quad (6, 5, 1)$

$x = 6,\ y = 5,\ z = 1 \qquad (6,5,1)$

$t = 2$

$x = 7,\ y = 9,\ z = -1 \qquad (7,9,-1)$

Ex. Let L be the line through $A(2,4,-3)$, $B(3,-1,1)$

At what point does this intersect the xy-plane? xy plan $z=0$

$\vec{v} = \overrightarrow{AB} = \langle 1,-5,4\rangle$

$A = \vec{r}_0 = \langle 2,4,-3\rangle$

symmetric equations: $\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$

sub $z=0$ into equation

$\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{3}{4}$

$x = \dfrac{11}{4}, \quad y = \dfrac{-15}{4} + \dfrac{16}{4} = \dfrac{1}{4}$

point $\boxed{\left(\frac{11}{4}, \frac{1}{4}, 0\right)}$

How do we describe the line segment $\overline{AB}$?

Parametric equations for L: $x = 2+t,\ y = 4-5t,\ z = -3+4t$

get poin A when $t=0$

get point B wen $t=1$

So $\overline{AB}$ is described by , restricting $0 \le t \le 1$

In general, take two points P_0, P_1 with position vectors $\vec{r}_0, \vec{r}_1$

Set $v = \vec{r}_1 - \vec{r}_0$

Vector equation: $r = r_0 + tv$

$= r_0 + t(r_1 - r_0)$

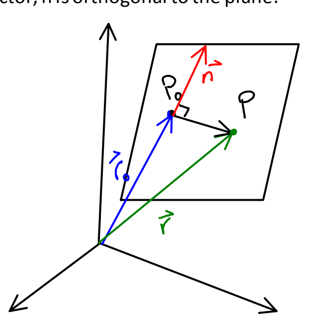
$= r_0 + tr_1 - tr_0$

$= (1-t)r_0 + tr_1 \quad 0<t<1 \qquad$ segment between r_0, r_1

Planes

We describe planes with a point P_0 and a normal vector $\vec{n}$

Normal vector, $\vec{n}$ is orthogonal to the plane.



$P(x,y,z)$

$P_0(x_0,y_0,z_0)$

$\vec{r} - \vec{r}_0 \perp \vec{n}$

$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$ vector eq. of a plane

$\vec{n} = \langle a,b,c\rangle$

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$ scalar equation of a plane

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$ scalar equation of a plane

Ex. Find an equation of the plane through (2,4,1) with normal vector $\vec{n}$ = <2,3,4>. Find its intercepts.

equation: $2(x-2)+3(y-4)+4(z+1)=0$

$2x+3y+4z-12=0$

any plane can be written in form $ax+by+cz+d=0$

x intercept

x-axis described by $y=z=0$

$2x-12=0 \quad x=6 \quad (6,0,0)$

y intercept

$3y-12=0 \quad (0,4,0)$

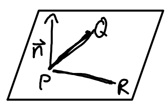
z intercept

$4z-12=0 \quad (0,0,3)$

Example: P(1,3,2) Q(3,01,6) R(5,2,0) Find the plane through these points.

$\vec{PQ} = \langle 2,-4,4\rangle$

$\vec{PR} = \langle 4,-1,-2\rangle$



$\vec{PQ}\times\vec{PR}$

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & -4 & 4 & 2 & -4 \\ 4 & -1 & -2 & 4 & -1 \end{array}\right| = 8i+16j-2k+16k+4i+4j$$

$$\vec{n} = 12i+20j+14k$$

Plane $\quad 12(x-1)+20(y-3)+14(z-2)=0$

*Note! Multiple possible answers. Can cross product any two PQ PR QR etc... can also choose P, Q, or R as point.

Example: find the angle between the planes.

x+y+z=1

x-2y+3z=1

*fact: The angle between the planes is the same as the angle between the normal vectors.

$\vec{n}_1 = \langle 1,1,1\rangle \qquad \vec{n}_2 = \langle 1,-2,3\rangle$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

In R^2, two lines L and M are either parallel or intersecting.

In R^3, two lines L and M are either parallel, intersecting, or skew

Example: Show that the lines are skew

L: x=1+t, y=-2+36, z=4-t

M: x=2s, y=3+s, z=-3+4s

Vectors associated to lines:

$L: \vec{v} = \langle 1,3,-1\rangle$

$M: \vec{w} = \langle 2,1,4\rangle$

Are these parallel? We could show that $\vec{v} \times \vec{w} = \vec{0}$
Or, notice that w is not a multiple of $\vec{v}$.

not parallel

Do they intersect?
Can we find t,s, that match up

solve $1+t = 2s \rightarrow t = 2s-1$

$-2+3t = 3+s \rightarrow -2+3(2s-1) = 3+s$

$-2+6s-3 = 3+s$

$5s = 8$

$s = 8/5$

$t = \frac{11}{5}$

check $4-t \neq -3+4s$

skew

Distance from a point to a plane:

$P(x_0, y_0, z_0)$

Plane $ax+by+cz+d=0$

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Distance from two parallel planes:
Just pick a point on one plane and apply distance formula.