Notes 14.2: Derivatives and Integrals of Vector Functions

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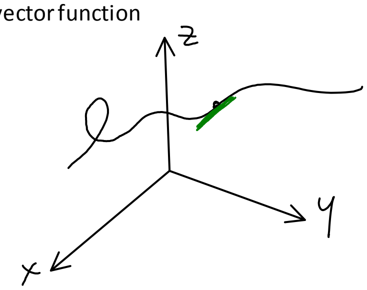
Derivatives and Integrals of Vector Functions

r(t) vector function



r'(t) gives a vector tangent to the space curve defined by r(t). The length of r'(t) is the sped at which the particle moves.

How to take a derivative of a vector function?

$r(t) = \langle f(t), g(t), h(t)\rangle$

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Example 1: Find r'(t) for $r(t)=\langle 1+t^3, te^{-t}, \sin(2t)\rangle$

$r'(t)=\langle 3t^2, -te^{-t}, 2\cos 2t\rangle$

Unit tangent vector: vector in the direction of r'(t) with length 1. $T(t) = \dfrac{\vec{r}\,'(t)}{|r'(t)|}$

Example 2: $r(t)=\langle 1+t^3, te^{-t}, \sin(2t)\rangle$ Find the unit tangent vector at the point where t=0.

$r'(0) = \langle 0, 1, 2\rangle$

$|r'(0)| = \sqrt{5}$

unit tangent vector $\left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

Tangent line at a point: line through $r(t_0)$ in the direction of $r'(t_0)$ or $T(t_0)$

Example 3: Find the tangent line through the point where t=0 as in examples 1 and 2.

$r(t)=\langle 1+t^3, te^{-t}, \sin(2t)\rangle$

Point: $r(0) = \langle 1,0,0\rangle$

Vector equation of tangent line:

$L(t) = \langle 1,0,0\rangle + t\langle 0,1,2\rangle$

Parametric equations:

x=1

y=t

z=2t

Example 4: Let $r(t) = \sqrt{t}\, i + (2-t) j$

- Find r'(t)
- Sketch r(1) and r'(1)

Find r'(t):

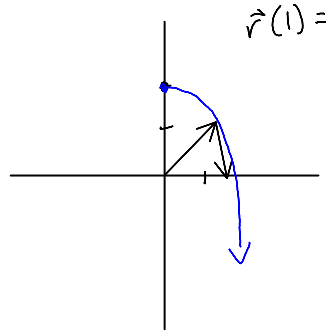
$\vec{r}(t) = \langle t^{1/2}, 2-t\rangle$

$\vec{r}\,'(t) = \left\langle \frac{1}{2\sqrt{t}}, -1\right\rangle$

Sketch:

Sketch:

$\vec{r}(1) = \langle 1, 1 \rangle \quad \vec{r}'(1) = \langle \frac{1}{2}, -1 \rangle$



Can we sketch r(t)?

$x = \sqrt{t}, \quad y = 2 - t$

$t = x^2 \quad y = 2 - x^2$

Example 5: Find parametric equations for the tangent line to the helix.

Helix: x=2cos(t), y=sin(t), z=t at (0,1,π/2)

$\sin^2 t + \cos^2 t = 1$

$\frac{x}{2} = \cos t$

$y^2 + \frac{x^2}{4} = 1 \quad$ *Ellipse*

Vector equation: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$

$\vec{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$

What value of t gives (0,1,π/2)

t=π/2

$\vec{r}'\left(\frac{\pi}{2}\right) = \langle -2, 0, 1 \rangle$

So, equations are:

$$\boxed{\begin{aligned} x &= -2t \\ y &= 1 \\ z &= \pi/2 + t \end{aligned}}$$

Second Derivative: r(t) as r''(t)

Definition: A curve given by a vector function r(t) on an interval I is called smooth if

- r(t) is continuous
- r'(t) ≠ $\vec{0}$ (is never 0 on I)

Example 6: Is the helix from example 5 smooth when I=R

- It is continuous
- Yes, never 0

  So it is smooth

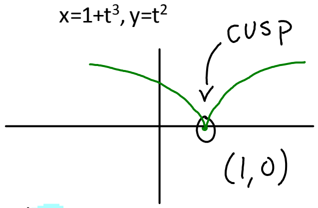
Example 7: Determine if r(t) = <1+t^3, t^2> is smooth.

- Yes, continuous
- r'(t) = <3t^2, 2t>

  Not smooth, when t=0 then r'(t)=<0,0>

  r(0)=<1,0>

  x=1+t^3, y=t^2



Differentiation Rules:

u, v = vector functions

c = scalar

f = real-valued function

f - real valued function

1) $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$

2) $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\frac{d}{dt}\vec{u}(t)$

3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f(t)\vec{u}\,'(t) + f'(t)\vec{u}(t)$

4) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}(t)\cdot\vec{v}\,'(t) + \vec{u}\,'(t)\cdot v(t)$

5) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}(t)\times\vec{v}\,'(t) + \vec{u}\,'(t)\times\vec{v}(t)$ *order important

6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}\,'(f(t))$ *chain rule

Integration of Vector Functions:

$\int \vec{r}(t)\,dt$ is defined by integrating components.

Example 8: $\vec{r}(t) = 2\cos t\, i + \sin t\, j + 2t\, k$

Find $\int_0^{\frac{\pi}{2}} \vec{r}(t)\,dt$

$$\int \vec{r}(t)\,dt = 2\sin t\, i - \cos t\, j + t^2 k \Big|_0^{\frac{\pi}{2}}$$

$$= \left(2i + 0j + \frac{\pi^2}{4}k\right) - \left(0i + j + 0k\right)$$

$$= 2i - j + \frac{\pi^2}{4}k$$ (vector)

Example 9: Show that if $|\vec{r}(t)| = C$ then r'(t) is orthogonal to r(t) for all t.

*NOTE: $\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$

$$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \frac{d}{dt}(c^2) = 0$$

$$\vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}\,'(t)\cdot\vec{r}(t)$$
$$2\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$
$$\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

So they are orthogonal for all t.