Notes 14.3: Arc Length and Curvature

Tuesday, July 10, 2007
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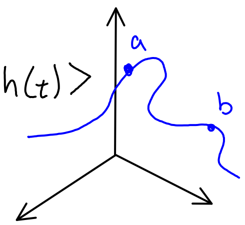
$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$



Distance along the curve from a to b is arc length

Arc Length Formula: The arc length of r(t) from t=a to t=b is given by:

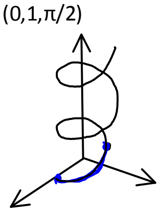
$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

☆Note: $\sqrt{f' + g' + h'} = |\vec{r}\,'(t)|$

so $L = \int_a^b |r'(t)|\, dt$

☆Note: these are hard

Example: Find the arc length along the helix r(t) = < cos(t), sin(t), t > from the point (1,0,0) to (0,1,π/2)



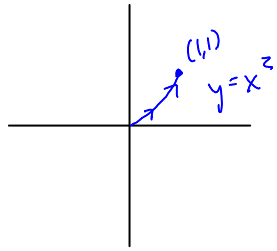
$f'(t) = -\sin t$
$(f'(t))^2 = \sin^2 t$
$g'(t) = \cos t$
$(g'(t))^2 = \cos^2 t$
$h'(t) = 1 = (h'(t))^2$

$$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$L = \int_a^b \sqrt{2}\, dt \qquad a = 0 \quad b = \frac{\pi}{2}$$

$$L = \int_0^{\pi/2} \sqrt{2}\, dt = \sqrt{2}\, t \Big|_0^{\pi/2} = \frac{\pi\sqrt{2}}{2} = \boxed{\frac{\pi}{\sqrt{2}}}$$

☆Note: a curve can be paramerterized in more than one way.



$\vec{r}_1(t) = \langle t, t^2 \rangle$
$0 \le t \le 1$
$\vec{r}_2(t) = \langle \sin t, \sin^2 t \rangle$
$0 \le t \le \frac{\pi}{2}$
$\vec{r}_3(t) = \langle t^3, t^6 \rangle$
$0 \le t \le 1$

We want to standardize the parameterization.
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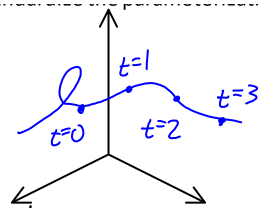
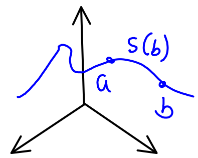


Reparametrizing with respect to arc length

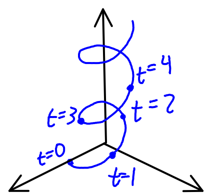
Definition: $s(t) = \int_a^t |r'(u)|\,du$

s(t) is the length between time a and time t.



Sometimes we can solve for t in terms of s in that formula. When we do that, we reparametrize r(t) as r(t(s))

Example: r(t) = < cos(t), sin(t), t > Reparametrize with respect to arc length in the increasing direction of t from (1,0,0).



$$s(t) = \int_a^t |r'(u)|\,du = \int_0^t \sqrt{2}\,du \Big|_0^t = t\sqrt{2}$$

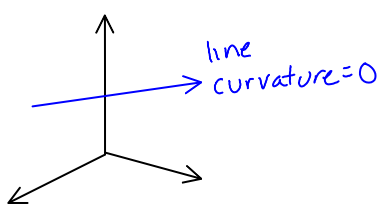
$$s(t) = t\sqrt{2}$$

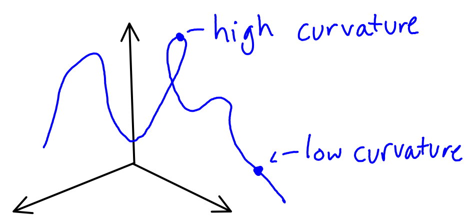
$$t = \frac{s(t)}{\sqrt{2}}$$

New parametrization:

$$\vec{r}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2} \rangle$$

Curvature:





We look at $\vec{T}(t) = \dfrac{r'(t)}{|\vec{r}\,'(t)|}$

curvature measures change in T(t)

Definition: $K = \left|\dfrac{dT}{ds}\right|$ after reparametrizing

Way around this:

By chain rule, $\dfrac{dT}{dt} = \dfrac{dT}{ds}\,\dfrac{ds}{dT}$

$$K = \left|\frac{dT}{ds}\right| = \left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right|$$

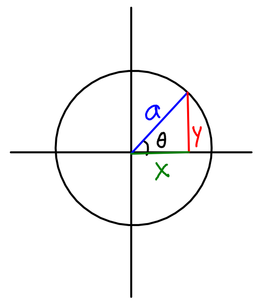
$s(t) = \int^b |r'(u)|du$

$$s(t) = \int_a^b |r'(u)|\,du$$

$$\frac{d}{dt} s(t) = |r'(t)|$$

So, $$K = \frac{|T'(t)|}{|r'(t)|}$$

Example: Find the curvature of a circle with radius a



$$\vec{r}(\theta) = \langle a\cos\theta,\ a\sin\theta \rangle$$

$$T(\theta) = \frac{\vec{r}\,'(\theta)}{|\vec{r}\,'(\theta)|}$$

$$\vec{r}\,'(\theta) = \langle -a\sin\theta,\ a\cos\theta \rangle$$

$$|r'(\theta)| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \sqrt{a^2(\sin^2\theta + \cos^2\theta)} = \sqrt{a^2} = a$$

$$T(\theta) = \frac{\langle -a\sin\theta,\ a\cos\theta \rangle}{a} = \langle -\sin\theta, \cos\theta \rangle$$

$$T'(\theta) = \langle -\cos\theta, -\sin\theta \rangle$$

$$|T'(\theta)| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\boxed{K = \frac{1}{a}}$$

Theorem: (3rd way to compute curvature)

$$K(t) = \frac{|\vec{r}\,'(t) \times r''(t)|}{(r'(t))^3}$$

✱ Note: order on this cross product does not matter because it is absolute value

Example: Find the curvature of y=f(x) (plane curve) r(x) = <x, f(x) >

$$\vec{r}\,'(x) = \langle 1, f'(x) \rangle = i + f'(x)j$$

$$\vec{r}\,''(x) = \langle 0, f''(x) \rangle = f''(x)j$$

$$\vec{r}\,'(x) \times \vec{r}\,''(x) = (i + f'(x)j) \times f''(x)j$$

$$= f''(x)(i \times j) + (f'(x)f''(x))(j \times j)$$

$$= f''(x)\,k$$

$$|\vec{r}\,'(x) \times r''(x)| = |f''(x)|$$

$$|\vec{r}\,'(x)| = \sqrt{1 + (f'(x))^2}$$

$$|\vec{r}\,'(x)| = \sqrt{1+(f'(x))^2}$$

$$K(x) = \frac{|f''(x)|}{\left(\sqrt{1+(f'(x))^2}\right)^3}$$

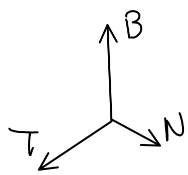
Normal and Binormal Vectors:

$|T(t)| = 1$ hence $T(t) \perp T'(t)$

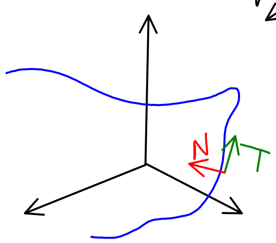
$T'(t)$ is not necesarilly a vector

Unit normal vector: $N(t) = \dfrac{T'(t)}{|T'(t)|}$

The binormal vector $B(t) = T(t) \times N(t)$



all orthogonal to each other

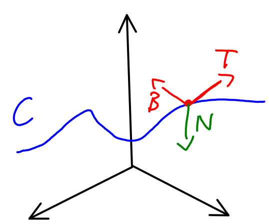


N points in direction of the curve

Definition:

1. The plane defined by N and B at a pint P on a curve C is called the normal plane of C at P.

2. The plane defined by T and N is called the osculating plane. (osculating means kissing)



3. .
   - The circle that lies in the osculating plane of C at P
   - has the same tangent vector at P as C
   - lies on the concave side of C
   - Has radius p=1/k is called the osculating circle.