Notes 14.4: Motion in Space: Velocity and Acceleration

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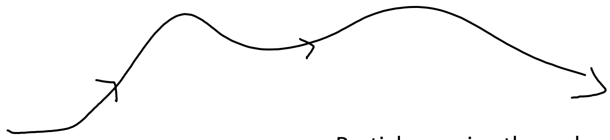


Particle moving through space position vector r(t).

Definition:

- The velocity vector $\vec{v}(t) = \vec{r}'(t)$.
- The acceleration vector is $a(t) = \vec{v}'(t) = \vec{r}''(t)$.
- The speed is the norm of the velocity vector. $v = |\vec{v}(t)|$.

Example 1: The position vector of an object is given by $r(t) = t^3 i + t^2 j$. Find $\vec{v}$, a, v at time t = 1

$$\vec{r}'(t) = 3t^2 i + 2t j = \vec{v}(t)$$
$$\vec{r}''(t) = 6t i + 2 j = \vec{a}(t)$$
$$v = \sqrt{9t^4 + 4t^2}$$

At $t = 1$:

$$\boxed{\begin{aligned} \vec{v}(1) &= \langle 3, 2 \rangle \\ \vec{a}(1) &= \langle 6, 2 \rangle \\ v &= \sqrt{13} \end{aligned}}$$

Example 2: A particle starts moving at position $\vec{r}(0) = \langle 1,0,0 \rangle$ with initial velocity $\vec{v}(0) = i - j + k$. Its acceleration is $\vec{a}(t) = 4ti + 6tj + k$. Find its position at time t.

$$\vec{v}(t) = \int \vec{a}(t)\,du$$
$$= \int 4ti + 6tj + k)\,du$$
$$= 2t^2 i + 3t^2 j + tk + \vec{C}$$

Plug in $\vec{0}$

$$\vec{C} = \vec{v}(0) = i - j + k$$

$$\boxed{\vec{v}(t) = \int \vec{a}(t)\,dt + \vec{v}(0)}$$

$$\vec{v}(t) = (2t^2 + 1)i + (3t^2 - 1)j + (t + 1)k$$

$$\boxed{\vec{r}(t) = \int \vec{v}(t) + \vec{r}(0)}$$

$$\vec{r}(t) = \left(\frac{2}{3}t^3 + t\right)i + (t^3 - t)j + \left(\frac{1}{2}t^2 + t\right)k$$

$+ i$

$$\left(\underline{2}\,t^2 + t + 1\right)i + (t^3 - t)j + \left(\underline{1}\,t^2 + t\right)k$$

$$\left(\frac{2}{3}t^2+t+1\right)i+\left(t^3-t\right)j+\left(\frac{1}{2}t^2+t\right)k$$

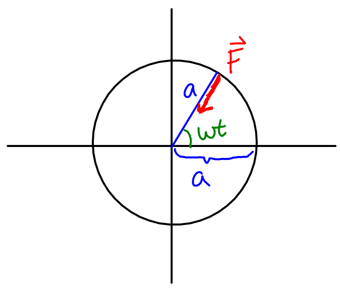
Newton's 3 Laws:

1. Inertia
2. F = ma (force = mass x acceleration) ← $\boxed{\vec{F}=m\vec{a} \quad \vec{F}(t)=m\vec{a}(t)}$
3. Equal and opposite force

Example 3: An object with a mass m that moves in a circular path with constant angular speed ω (omega) has position vector $\vec{r}(t)=a\cos(wt)i+a\sin(wt)j$
Find the force acting on the object.

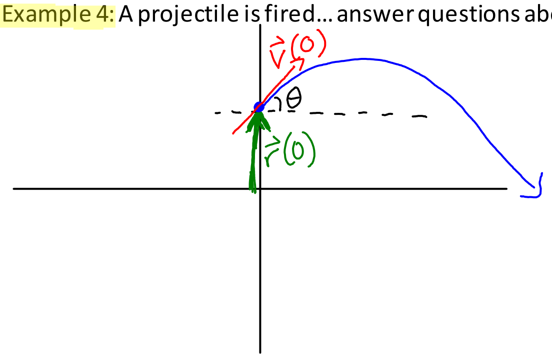
$$\vec{v}(t)=-aw\sin(wt)i+aw\cos(wt)j$$
$$\vec{a}(t)=-aw^2\cos(wt)i-aw^2\sin(wt)j$$
$$\vec{F}(t)=m\vec{a}(t)=-maw^2(\cos(wt)i+\sin(wt)j)$$

Which direction is the force pointing?



Force at time t is pointing toward the origin.
(Centripetal force)

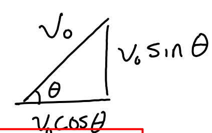
Example 4: A projectile is fired... answer questions about it.



$$\vec{a}(t)=-gj$$
$$\vec{v}(t)=-gtj+\vec{v}(0)$$
$$\vec{r}(t)=-\frac{1}{2}gt^2j+\vec{v}(0)t+\vec{r}(0)$$

$|\vec{v}(0)|=v_0$ (initial speed

$\vec{v}(0)=v_0\cos\theta i+v_0\sin\theta$



$$\boxed{\begin{aligned}\vec{v}(t)&=v_0\cos\theta i+(v_0\cos\theta-gt)j\\ \vec{r}(t)&=v_0t\cos\theta i+\left(v_0t\sin-\frac{1}{2}gt^2+h\right)j\end{aligned}}$$

Examples:

- Speed when the particle hits the ground?
- What is the maximum elevation of trajectory?
  - When y velocity = 0

Tangential and Normal Components of Acceleration:

We look more closely at the acceleration vector $\vec{a}(t)$.

Tangent Vector T(t) $= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{v}$ so $\vec{v}(t) = v\,T(t)$

Take derivatives:

$$\boxed{*\ \ \vec{a}(t) = v'\,T(t) + v\,T'(t)}$$

<u>Recall:</u> Curvature $K = \frac{|T'|}{|r'|} = \frac{|T'|}{v}$ so $|T'| = vK$

$$N = \frac{T'}{|T'|} = \frac{T'}{vK} \Rightarrow vKN$$

$$\boxed{*\ \text{becomes}\ \ \vec{a} = \underbrace{v'}_{a_T,\ \text{Tangential component}}\,T + \overbrace{v^2K}^{a_N,\ \text{normal component}}\,N}$$

Acceleration occurs in the osculating plane.

Let's express a_T, a_N, in terms of $\vec{r}$, $\vec{r}\,'$, $\vec{r}\,''$.

To do this, note:

$$\vec{v}\cdot\vec{a} = vT\cdot(v'T + v^2KN)$$

$$= vv'\underbrace{(T\cdot T)}_{T\text{ is a unit vector}} + (v^3K)\cancelto{0}{(T\cdot N)} \quad \text{as } T \perp N$$

$$= vv'$$

So, $\boxed{a_T = v' = \frac{\vec{v}\cdot\vec{a}}{v} = \frac{\vec{r}\,'\cdot\vec{r}\,''}{|\vec{r}\,'|}}$

$$a_N = v^2K = |\vec{r}\,'|^2\,\frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$\boxed{a_N = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|}}$$

Example 5: A particle moves with position function r(t) = $\langle t^2, t^2, t^3\rangle$. Find a_T, a_N.

$$\vec{r}\,'(t) = \langle 2t, 2t, 3t^2\rangle$$

$$\vec{r}\,''(t) = \langle 2, 2, 6t\rangle$$

$$a_T = \frac{|\vec{r}\,'\cdot r''|}{\ } = \frac{4t + 4t + 18t^3}{\ } = \boxed{8t + 18t^3}$$

$|\vec{r}\,'| \quad \sqrt{4t^2+4t^2+9t^4} \quad \boxed{\sqrt{8t^2+9t^4}}$