Notes: 15.2 Part 1: Limits

Wednesday, July 18, 2007
1:36 PM

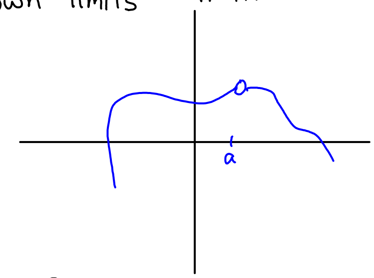
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ does not exist}$$

In general we write $\lim_{(x,y)\to(a,b)} f(x,y) = L$

$f(x,y) \to L$ as $(x,y)\to(a,b)$

Known limits: in $\mathbb{R}^2$

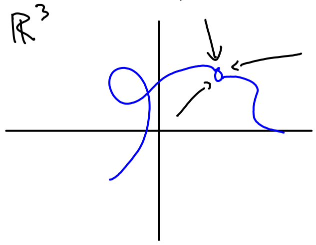


What is $\lim_{x\to a} f(x)$?

We look at $\lim_{x\to a^+} f(x) \overset{?}{=} \lim_{x\to a^-} f(x)$

limit exists if these are equal

In $\mathbb{R}^3$



Lots of paths to this point. We look at limits along paths.

Fact: If $f(x,y)\to L_1$ along path C_1 and $f(x,y)\to L_2$ along C_2 where $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist

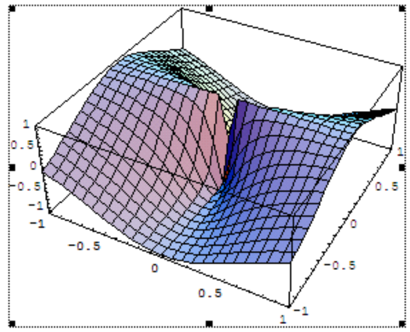
Example 1: $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ as $(x,y)\to(0,0)$

What happens as we approach (0,0) along the y-axis

y-axis is $x=0$

$f(x,y) = \frac{-y^2}{y^2}$ along y axis

$\lim_{y\to 0} \frac{-y^2}{y^2} = -1$



How about the x-axis?

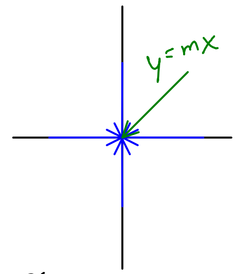
$f(x,y) = \frac{x^2}{x^2} \xrightarrow{x\to 0} 1$

So this limit does not exist.

*Caution: Checking a lot of paths can't prove the limit exists. Proving the limit does exist is harder.

Example 2: $\lim \quad xy$

$(x,y)\to(0,0) \quad \overline{x^2+y^2}$



- x axis $(y=0) \Rightarrow \lim_{x\to 0} \frac{0}{x^2} = 0$
- y axis $(x=0) \Rightarrow \lim_{y\to 0} \frac{0}{y^2} = 0$
- along $y=mx \Rightarrow \lim_{x\to 0} \frac{x\cdot mx}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m\cancel{x^2}}{(1+m^2)\cancel{x^2}} = \lim_{x\to 0} \frac{m}{1+m^2}$

let $m=1 \Rightarrow \lim = \frac{1}{2}$

let $m=2 \Rightarrow \lim = \frac{2}{5}$

$\boxed{\text{limit does not exist}}$

In general, if you do y=mx, and the answer depends on m, the limit does not exist.

Example 3: $f(x,y) = \frac{xy^2}{x^2+y^4}$ as $(x,y)\to(0,0)$

- $y=mx$

$$\lim_{x\to 0} \frac{x\cdot m^2\cdot x^2}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2(1+m^4x^4)} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$$

Is there either x = ? or y = ? So that the degree of the top equals the degree of the bottom.

for $\frac{xy^2}{x^2+y^4}$ $\quad x = my^2$

$$\lim_{y\to 0} \frac{my^4}{m^2y^4+y^4} = \frac{my^4}{(1+m^2)y^4} = \frac{m}{1+m^2}$$

$\boxed{\text{lim DNE}}$

Example 4: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

- $y=mx \quad \lim_{x\to 0} \frac{3x^3m}{x^2+m^2x^2} = \frac{3xm}{m^2} = \frac{3x}{m} = 0$

- $y=mx^2 \quad \lim_{x\to 0} \frac{3x^2(mx^2)}{x^2+m^2x^4} = \lim_{x\to 0} \frac{3mx^4}{x^2(1+mx^2)} = \lim_{x\to 0} \frac{3mx^2}{1+m^2x^2} = 0$

Try to prove this limit exists:

<u>Squeeze Theorem:</u> If $g(x,y) \le f(x,y) \le h(x,y)$ near (a,b) and $g(x,y)\to L$ as $(x,y)\to(a,b)$ and $h(x,y)\to L$ as $(x,y)\to(a,b) \Rightarrow f(x,y)\to L$ as $(x,y)\to(a,b)$.

<u>Special Case:</u> If you're trying to prove $L=0$ then look at $|f(x,y)|$. Find $h(x,y) \ge |f(x,y)|$ and prove $h(x,y)\to 0$ as $(x,y)\to(a,b)$

Why? If $|f(x,y)|\to 0$, then $f(x,y)\to 0$ but $0 \le |f(x,y)|$ in any case.

Example 4 continued: Prove $\frac{3x^2y}{x^2+y^2} \to 0$

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$$0 \le \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} = \left(\frac{x^2}{x^2+y^2}\right)3|y| \le 3|y|$$

$$\underset{\downarrow \atop 0}{0} \le \left|\frac{3x^2y}{x^2+y^2}\right| \le \underset{\downarrow \atop 0}{3|y|}$$

$\boxed{\text{so by squeeze theorem } \lim_{y\to 0} \left|\frac{3x^2y}{x^2+y^2}\right| \to 0}$

Example 5: $\lim_{(x,y)\to(0,0)} \frac{x^8y^4}{x^2+y^2}$

$$\underset{\downarrow \atop 0}{0} \le \frac{x^8y^4}{x^2+y^2} \le \frac{x^8y^4}{x^2} = \underset{\downarrow \atop 0}{x^6y^4}$$

$\boxed{\text{so by squeeze theorem} \to 0}$