Notes 15.3 Part II

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Partial Derivatives of Functions of More than 2 Variables:

If f is a function of 3 variables, say f(x,y,z) $f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$

Defined similarly for f_y, f_z.

If f is a function of n variables $f(x_1,x_2,\ldots,x_n)$

$$f_{x_i} = \lim_{h \to 0} \frac{f(x_1, x_i+h, x_n) - f(x_1, x_i, x_n)}{h}$$

Example: Find f_x, f_y, f_z if $f(x,y,z) = e^{xy}\ln z$

$$f_x = (\ln z)\, y e^{xy}$$
$$f_y = (\ln z)\, x e^{xy}$$
$$f_z = (e^{xy}) \cdot \frac{1}{z} = \frac{e^{xy}}{z}$$

$f_x(x,y,z)$ is a function of 3 variables so we can take its partial derivatives. These (along with the partials of f_y, f_z) are called the second partials of f.

Q: How may second partials does a function of 3 variables have?
A: Nine
Q: How many nth partials?
A: 3^n

If f is a function of 2 variables, it has 2^n nth partials.

Notation:
$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ (take partial with respect to x first and then y)

Example: Find all 2nd partials of $f(x,y) = 2x^2y + e^{xy} - 3y$

$$f_x = 4xy + ye^{xy} \qquad f_y = 2x^2 + xe^{xy} - 3$$

$$f_{xx} = 4y + y^2 e^{xy} \qquad f_{yy} = x^2 e^{xy}$$

$$f_{xy} = 4x + yxe^{xy} + e^{xy} = f_{yx} = 4x + xye^{xy} + e^{xy}$$

Clairault's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If f_{xy}, f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$. i.e. if f is "nice" then $f_{xy}=f_{yx}$.

Clairault's Theorem also applies to higher partial derivatives. i.e. if f is a function of 3 variables, then $f_{xxyzxy} = f_{xxxyyz} = \ldots$

Q: How many partials does f(x,y) have?
A: 3
Q: How many 3rd partials?
A: f_{xxx}, f_{xxy} f_{xyy} f_{yyy} = 4
Q: How many nth partials?
A: n+1
Q: How many 2nd partials does f(x,y,z) have?
A: f_{xy}, f_{xz}, f_{yz}, f_{xx}, f_{yy}, f_{zz}, 6
Q: How many 3rd partials?
A: $f_{\_\_\_}$
Answer next class.

Example: Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$.

$$f_x = \cos(3x+yz)\cdot(3) = 3\cos(3x+yz)$$
$$f_{xx} = -3\sin(3x+yz)\cdot 3 = -9\sin(3x+yz)$$
$$f_{xxy} = -9\cos(3x+yz)\cdot(z) = -9z\cos(3x+yz)$$
$$f_{xxyz} = -9\left(z\cdot -\sin(3x+yz)\cdot y + \cos(3x+yz)\right)$$
$$= 9yz\sin(3x+yz) - 9\cos(3x+yz)$$

$f_{xxyz} = 9yzf(x,y,z) - 3f_x(x,y,z)$ so f satisfies this partial differential equation.

Usually, we denote the function as U.
$U_{xxyz} = 9yzU - 3U_x$

2 questions you can ask for partial differential equations:

1. Given a function, does it satisfy the partial differential equation.
2. Given this equation, find a function that satisfies it.

Example: Laplace's Equation. Show that $u(x,y) = e^x\sin y$ is a harmonic function.

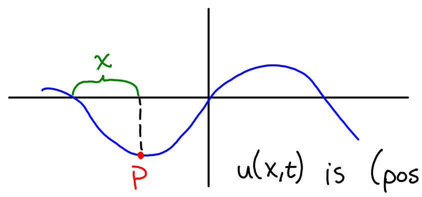
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$$

Solutions of this equation are called harmonic functions.

$$u_x = e^x\sin y \qquad u_y = e^x\cos y$$
$$u_{xx} = e^x\sin y \qquad u_{yy} = -e^x\sin y$$

$u_{xx} + u_{yy} = 0$ so function is harmonic

We also have (for $u(x,t)$) $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ for some a = real number. This is called the wave equation.



u(x,t) is (pos. or neg.) distance from the line at x units from the node at time t.

Example: $u(x,t) = \sin(x-at)$ Show that u satisfies the wave equation.

$$u_t = \cos(x-at)(-a) = -a\cos(x-at)$$
$$u_{tt} = -a(-\sin(x-at))(-a) = -a^2\sin(x-at)$$
$$u_x = \cos(x-at)$$
$$u_{xx} = -\sin(x-at)$$
$$u_{tt} = a^2u_{xx}$$

Example: $u(x,t) = (x-at)^6 + (x+at)^6$ Show that this equation satisfies the wave equation.

$$u_x = 6(x-at)^5 + 6(x+at)^5$$
$$u_{xx} = 30(x-at)^4 + 30(x+at)^4$$

$$u_t = 6(x-at)^5(-a) + 6(x+at)^5(a)$$
$$= 6a(-(x-at)^5 + (x+at)^5)$$
$$u_{tt} = 6a[-5(x-at)^4(-a) + 5(x+at)^4a]$$
$$= 30a^2[(x-at)^4 + (x+at)^4]$$

$u_{tt} = a^2u_{xx}$ so this satisfies the wave equation.

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