Notes: 15.4 Part I

Wednesday, July 25, 2007
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Q: $f(x,y,z)$ a "nice" function of 3 variables. How many 3rd partial derivatives does it have?
A: $f_{\_\,\_\,\_}$

"stars + bars" method

$\underline{x}\ \underline{x}\ |\ \underline{y}\ |$

$|\ *\ |\ *\ \ * = yzz$

How many ways are there to line up 3 stars, 2 bars?

$\underline{|}\ \underline{*}\ \underline{|}\ \underline{*}\ \underline{*}$

5 blanks, each one gets * or I. We're choosing the 3 places the stars go.
How many ways are there to choose 3 objects from 5.

$$\binom{5}{3} = 10$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$f(x_1, \dots, x_n)$

How many k-th partial derivatives?

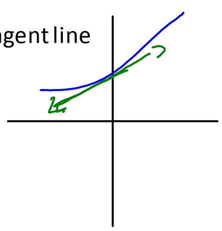
\# of *

We need n-1 bars.
In total, n-1 + k blanks. We want to choose k

$$\binom{n-1+k}{k}$$

Tangent Planes:

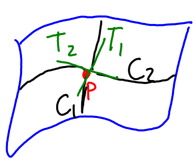
One variable: tangent line



As we zoom in on the curve, the curve becomes similar to the tangent line.

In 2 variables: we have the tangent plane.

Take a surface S given by $z=f(x,y)$. Let $P(x_0,y_0z_0)$ be a point on S. Assume f has continuous first partial derivatives.



Let C_1, C_2 be the curves on S through P, parallel to the coordinate axis.
Let T_1, T_2 be the tangent lines to C_1, C_2 through P. The plane defined by those 2 lines is the <u>tangent plane.</u>

Let's find the equation of the tangent plane. Any plane through P is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Let's assume C is not 0 so we divide by -C.

$a(x - x_0) + b(y - y_0) = (z - z_0)$

Q: What is the intersection of the tangent plane with $y = y_0$?
A: The line T_1

T_1 has equation.

Slope: $f_x(x_0,y_0)$

$z=f(x_0,y_0)(x - x_0)$

$a(x - x_0) = z - z_0$ is the intersection of the tangent plane with $y=y_0$. So $a = f_x(x_0,y_0)$.

Similarly, $b=f_y(x_0,y_0)$.

So the equation of the tangent plane to the surface $z = f(x,y)$ at $P(x_0,y_0,z_0)$ is $z - z_0 = f_x(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$
Note: $z_0 = f(x_0,y_0)$

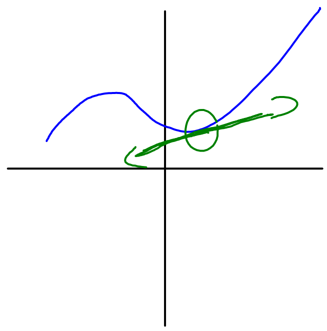
Example: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at (1,1,3).

$f(x,y) = 2x^2+y^2$

$f_x = 4x \qquad f_y = 2y$

$\boxed{z - 3 = 4(x-1) + 2(y-1)}$

Linear Approximations:



Recall: Curve similar to tangent line near the point of tangency.

Our last tangent plane was $z - 3 = 4(x-1) + 2(y-1)$
$= 4x - 4 + 2y -2$
$= 4x + 2y -6$
$z= 4x + 2y - 3$
$L(x,y)$

Idea: near $(x,y) = (1,1)$, $L(x,y) \approx f(x,y)$

e.g. (1.1, 0.95)
L(1.1, .095) = 3.3
f(1.1, 0.95) = 3.225

How do we get linear approximations in general?

In general the tangent plane is given by $z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$.

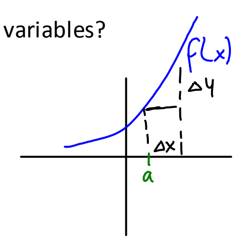
So $z = L(x,y) = f_x(x_0,y_0)(x - x_0) + f(x_0,y_0)(y - y_0) + f(x_0,y_0)$

*Notation: This is also called the tangent plane approximation.

Recall: We only defined tangent planes for f_x, f_y continuous.

Q: What is the notion of differentiability for functions of two variables?

Function f(x) of one variable. Δx, Δy
Δx can be anything (independent variable)
Δy = f(a+Δx) - f(a)



If f is differentiable at a, Dy = f'(a) Δx + EΔx
Where E -> 0 as Dx ->0

How to do this for 2 variables? We setup Δx, Δy (increments) both are independent variables.
Δz (total increment) = f(a + Δx), b + Δy) - f(a,b)

A: We say f(x,y) is differentiable if $Dz = f_x(a,b)\Delta x + f_y(a,b)\Delta y + E_1\Delta x + E_2\Delta y$ where E_1, E_2 ->0 as (Δx,Δy) -> (0,0)

Theorem: If the partial derivatives f_x, f_y are continuous at a point (a,b), then the function is differentiable at that point.
Useful for proving something is differentiable but cannot prove that something is not differentiable.

Q: $f(x,y)$ is differentiable at (a,b) Can we conclude its first partials are continuous.
A: No

Example: $f(x,y) = x^2 + y^2$ Show that f is differentiable at $(1,1)$.

$$f_x = 2x \qquad f_y = 2y$$

Both are polynomials so they are both continuous, so f is differentiable.